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Distinguishable- and Indistinguishable-Particle Descriptions of Systems of Identical Particles

W. M. DE MUYNCK

Department of Physics, Eindhoven University of Technology, Eindhoven, the Netherlands

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Abstract

The problem of distinguishability of identical particles is considered from both experimental and theoretical points of view. It is argued that distinguishability has to be defined relative to a definite set of experiments and that the criterion by which the particles are distinguished should be specified. Failure to do so may cause mismatching between theory and experiment. On the theoretical level a distinction is made between indexed- and unindexed-particle theories, indices being unobserved intrinsic properties of the particles. A field theory of indexed particles is constructed and shown to be equivalent to the second quantization formalism, which is an unindexed-particle theory.

1. *Introduction*

It was observed by Mirman (1973) that the concepts of "identical particles" and "distinguishability" or "indistinguishability" have been poorly formulated in quantum theory. In most textbooks these notions are treated in a rather vague and intuitive way, without a firm relation to experiment. More or less the same remarks apply to the relation between (in)distinguishability and the symmetry of the wave function of a many-particle system ("statistics").

Let us start with the notion of "identical particles." Most definitions are rather tautological, like the one given by Messiah (1965): Two particles are identical if their physical properties are exactly the same. Strictly speaking, this definition would infer that two particles never can be identical, unless position is considered no longer as a physical property. The distinction made by Jauch (1966) between intrinsic properties (i.e., properties that are independent of the state of the system) and extrinsic properties (depending on the state of the system) seems to be more promising in this respect. Jauch defines two (elementary) particles to be identical if they agree in all their *intrinsic* properties. Position, as a dynamical variable, is an extrinsic property. So Messiah's problem does not arise any more.

The distinction between intrinsic and extrinsic properties also introduces the possibility of revealing the dependence of the identity of particles on the experimental and theoretical context. Thus, by enlarging the class of experiments considered, there are two causes by which the identity of two particles may change: Either a new intrinsic property may be discovered, showing the difference of two particles that were thought to be identical; or an (up until now) intrinsic property may show up dynamical behavior, thus making two different particles identical (when the other intrinsic properties are the same). We shall have reason to change Jauch's definition slightly in such a way that this dependence of the identity of particles on the class of experiments considered is more explicit. As this has to do with (in)distinguishability of the particles, we consider this next.

As to (in)distinguishability, several different opinions can be found in the literature, e.g., (a) in classical as well as in quantum mechanics identical particles are indistinguishable (Messiah, 1965), (b) identical particles are distinguishable classically but indistinguishable quantum mechanically (Jauch, 1966), (c) same as (b) when the wave packets of the particles are overlapping; but identical particles are distinguishable also quantum mechanically when they are far apart or when there is some constant of the motion by which the particles can be distinguished (Schiff, 1955), (d) classically as well as quantum mechanically identical particles are distinguishable whenever the *histories* of the particles experimentally allow a distinguishing criterion (Mirman, 1973).

Whereas Messiah (a) seems to use the concepts of identity and indistinguishability of particles in a tautological manner, the others go beyond this by trying to find a way to label or name the identical particles, thus trying to give each particle a kind of individuality that is maintained throughout time. Cases (b), (c), and (d) have in common that the labeling is thought to be performed by a dynamical variable or extrinsic property, the intrinsic properties of identical particles being the same. In classical mechanics generally the position variable will do, although Mirman notices some reservations also here, stemming from experimental inability to follow the continuous paths of particles when they are very close together. In quantum mechanics the absence of particle trajectories makes the position variable inappropriate for distinguishing purposes whenever the wave functions of the two particles are overlapping. This leads Jauch to the rigorous conclusion of indistinguishability, whereas Schiff points out some exceptions to this rule. According to the latter, two identical particles are distinguishable when the two-particle probability amplitude $a(1, 2)$ of some dynamical variable is different from zero only when the two particles have their values in *disjoint* ranges of the spectrum of the variable. Note, however, that this can never occur when the wave function is (anti)symmetric, for then $a(2, 1) = \pm a(1, 2)$.

In order to analyze the experimental meaning of the concepts of identical particles and distinguishability Mirman (1973) considers two electrons in different galaxies. A more common paradigm could be provided by the situation of two identical particles crossing a bubble chamber at the same moment without colliding. When the particle energies are high enough we see two completely separated and distinguishable tracks. With Mirman these data seem to be sufficient to speak of two distinguishable particles: The particles are distinguished by their tracks. However, when the wave function is (anti)symmetric, this is clearly in conflict with Schiff's view, because in that case we cannot tell which of the two particles is in which of the two tracks. According to Schiff we could only speak of distinguishable particles when there is a one-to-one correspondence between each of the tracks and one of the indices 1 and 2 of the two particles. It is clear from this that studying the experimental meaning of distinguishability amounts to studying the experimental meaning of the indices 1 and 2 of the two particles. This can be done for classical and quantum mechanics simultaneously, because in the elementary forms of both theories these indices obtain.

Since an index is thought to be independent of the state of the particle, it is to be considered as an intrinsic property. As a consequence of this, according to Jauch's definition particles that have different indices are not identical. By this reasoning the problem of distinguishability is defined away, because nonidentical particles being distinguishable, this leads to the conclusion that the particles described by elementary classical and quantum mechanics are distinguishable. However, this statement follows from purely theoretical considerations. So it tells more about the form of the theories than about the experimental meaning of the indices. The only conclusion we can draw is that the *languages* of elementary classical and quantum mechanics are those of *distinguishable* particles. Such languages are bound to be not very adequate to describe particles that are indistinguishable experimentally, which may be the reason for the different views on distinguishability found in literature. These languages would only be adequate when detectors were existing that are sensible to the indices. When this is not the case it is preferable to use languages that are independent of the indices, i.e. in which dynamical variables without experimental meaning like $\mathbf{r}_j$, the position vector of particle j, do not show up. Such languages do exist for quantum as well as classical mechanics: quantum field theory (second quantization) and the classical limit of the field theoretic approach to statistical mechanics. Quantities that do have experimental meanings for particles without indices are, for instance, the probability $\rho(\mathbf{r}_a)d\mathbf{r}_a$ of finding a particle in the volume element $d\mathbf{r}_a$ near the space point $\mathbf{r}_a$, or correlation functions like $\rho(\mathbf{r}_a, \mathbf{r}_b)$ correlating the probabilities of finding particles in space points $\mathbf{r}_a$ and $\mathbf{r}_b$. It is essential that the indices a and b do not refer to particles here, but to detectors at space points $\mathbf{r}_a$ and $\mathbf{r}_b$. In this language the only relevant question is whether a detector has counted a particle or not, particle indices playing no role. Distinguishability now reduces to Mirman's purely experimental notion (Mirman, 1973), expressed by, e.g., the mutual distance of the two detector positions $\mathbf{r}_a$ and $\mathbf{r}_b$ and the conduct of $\rho(\mathbf{r}_a, \mathbf{r}_b)$ as a function of these parameters. This point of view seems to be consistent with the method Lyuboshitz & Podgoretskii (1969, 1971) use to treat the interference of nonidentical particles.

2. *Experimental Relevance of Indices*

Apart from particle creation and annihilation, elementary quantum mechanics and second quantization are equivalent theories (Heisenberg, 1930). Also the classical theories of particles with and without indices are equivalent when only results are considered that are independent of the particle indices.

As we saw, indices are to be considered as intrinsic properties. Then, using Jauch's definition of identical particles, particles that are identical in the un-indexed version of the theory are different in the indexed one. This is undesirable because the theories describe the same set of experiments of the same set of particles. To overcome this difficulty we change the definition of identical particles slightly, making use of the observation that apparently the index does not have influence on the dynamical behavior of the particles. So particles are identical in both theories when we change the definition in the following sense:[1]

> Two particles are identical if they agree in all their *dynamically relevant* intrinsic properties.

This definition stresses the relevance of the set of experiments considered, because intrinsic properties that are dynamically irrelevant with respect to some set of experiments may become dynamically relevant on extending this set. For instance, consider the set of all collision experiments performed with two billiard balls that differ only in color. Then color is a dynamically irrelevant intrinsic property and as such may serve as an index. The two balls are identical with respect to the set of collision experiments. However, when we extend the set of experiments to include also absorption and reflection of light, the two balls are no longer identical, for they will show up different dynamical behavior in absorbing and reflecting light.

Besides showing the dependence of the notion of "identical particles" on the experimental setting, the example reveals another aspect pertaining to the experimental meaning of distinguishing identical particles by indices: When particles are identical with respect to some set of experiments they can only be distinguished by an index when it is possible to extend the set of experiments in such a way that the index becomes dynamically relevant with respect to some new property. When such extension is impossible or unknown the index does not have any experimental relevance in distinguishing the particles. In this case only Mirman's criterion remains. This seems to be the situation in elementary particle physics. However, this is not a reason to deny all physical relevance to the indices of the particle variables obtaining in elementary quantum mechanics. As we saw, distinguishability by means of indices is based on incompleteness of the description of the dynamical behavior of the particles. Then, the mere possibility of an indexed-particle language is an indication that there might be some new field of research connected with these indices.

[1] It will turn out to be essential that the definition also concerns particles that are not elementary.

Some evidence in this direction could be provided by the problem of a dynamical description of the process of (anti)symmetrization of the wave function when two identical particles approach each other from great mutual distance (Mirman, 1973). As Mirman stipulates, when two identical particles are created simultaneously at great distance from each other, immediate (anti)symmetry of the two-particle wave function would violate causality. This shows the necessity for studying this (anti)symmetrization process, which cannot be done using existing quantum field theories, because these are constructed as causal theories that do correspond to immediate (anti)symmetry of the wave function.

There exists at least one physical situation in which different symmetries of the wave functions of initial and final states are necessary to describe experimental evidence. It is not surprising that this situation, viz., that illustrating the famous Gibbs paradox, has been connected with the problem of distinguishability of identical particles for quite a long time (e.g., Schrödinger, 1957). One considers an ideal gas of N identical particles in volume V, initially divided by an impermeable diaphragm in two parts A and B having the same temperature and pressure. When the diaphragm is removed, entropy, being an extensive property, should behave in such a way that the entropy of the combined final state should equal the sum of the entropies of parts A and B in the initial state. From the quantum mechanical definition of entropy

$$S = -k \operatorname{Tr} \rho \ln \rho \qquad (2.1)$$

resulting in the Sackur–Tetrode expression (Schrödinger, 1957) for systems in thermal equilibrium, it can be seen that this is satisfied when the density operator of the initial state obeys

$$\rho = \rho_A \rho_B \qquad (2.2)$$

whereas the final state should be totally (anti)symmetrized. The density operators ρ_A and ρ_B are thought to describe the subsystems A and B and correspond to (anti)symmetrized states of these subsystems separately.

When the particles of A are different from those of B a diffusion process is started by the removal of the diaphragm, resulting in a homogenous mixture of the two gases. When the particles in A and B are identical this diffusion process is often considered to be unnoticeable or even unreal (Schrödinger, 1957). This paradoxical situation might be resolved when the particles are indexed particles. Then the diffusion process is as real as in the case of different particles. Moreover, when the (anti)symmetrization process is considered as a description of this diffusion process, it is not even unnoticeable since it clearly substantially influences the state of the total system.

The relation to distinguishability by means of indices comes into play through equation (2.2). Here A and B are indices, not of particles, but of assemblies of particles (they correspond to particles when A and B contain one particle each). The possibility of introducing density operators of well-defined subsystems A and B in the theoretical language is clearly related to the fact that the particles of subsystem A are distinguishable experimentally from those of subsystem B as long as the impermeable diaphragm is present. So A and B are distinguishing indices, corresponding to the regions to which the subsystems A and B are confined experimentally in the initial state.

In the indexed language

$$\rho_A = \mathrm{Tr}_B \rho, \quad \rho_B = \mathrm{Tr}_A \rho \tag{2.3}$$

where Tr_A, Tr_B means partial tracing over the particles contained in A, B. The question of distinguishability of the two subsystems by the indices A and B after removal of the diaphragm is connected with the feasibility of expressions (2.3) in this latter situation. It is often stated that there exists a causal relationship between indistinguishability and (anti)symmetry of the wave function. If this were true, the indices A and B would loose their meaning in the process of (anti)symmetrization. However, we will see in the next section that the causal relationship mentioned above does not exist. This will remove (anti)symmetry of the wave function as an a priori obstacle to distinguishability of the two subsystems by means of the indices A and B, also in the final state.

The experimental distinguishability of systems A and B by the indices A and B in the initial state is based on the one-to-one correspondence of some intrinsic property of the systems with positions of the particles. This correspondence is lost in the final state. Because in the diffusion process experimental distinguishability in the sense of Mirman by, e.g., the positions of the particles breaks down, only the yet unknown property is left to distinguish A and B. It is possible to ignore this property and to leave it completely outside the theoretical language. However, in acting this way we may neglect some phenomena that seem to be not devoid of physical relevance.

3. *Indistinguishability and "Statistics"*

As we saw in Sec. 1 the language of the elementary wave mechanics of wave functions[2] $\psi(x_1, x_2, \ldots, x_n)$ is a language of distinguishable particles. Although it is not impossible to stick to this description when the indices are thought to be physically irrelevant, some extra care is needed. One instance where this becomes apparent is provided by the attempt to derive (anti)symmetry of the wave function ("statistics") from indistinguishability, made in some textbooks (e.g., Landau & Lifshitz, 1965; Blokhintsev, 1964). The reasoning is as follows:

Consider the expectation value[3] of, e.g.[4] the observable $f(x_1, \ldots, x_n)$,

$$\langle f(x_1, \ldots, x_n)\rangle = \int dx_1 \ldots \int dx_n f(x_1, \ldots, x_n) |\psi(x_1, \ldots, x_n)|^2 \tag{3.1}$$

Becasue of indistinguishability of the particles the following equality is postulated[4]:

$$\langle f(x_1, \ldots, x_n)\rangle = \langle f(P(x_1, \ldots, x_n))\rangle \tag{3.2}$$

[2] $x_j = (\mathbf{r}_j, \mu_j)$, μ_j being the spin variable of particle j.
[3] $\int dx_j$ stands for integration over the space variable and summation over the spin variable.
[4] The same should be done with functions of momentum or any other one-particle observable.

$P(x_1, \ldots, x_n)$ being an arbitrary permutation of $(x_1, \ldots, x_n)$. When (3.2) should hold for any function f, the equality inevitably leads to

$$|\psi(P(x_1, \ldots, x_n))|^2 = |\psi(x_1, \ldots, x_n)|^2 \tag{3.3}$$

which implies symmetry or antisymmetry of the wave function.

But as the particles are presupposed to be indistinguishable (at least by way of indices) not every observable $f(x_1, \ldots, x_n)$ has physical significance. Only those observables that have a counterpart in the unindexed theory should be considered in (3.2). This being precisely the class of totally symmetric functions for which

$$f(P(x_1, \ldots, x_n)) = f(x_1, \ldots, x_n) \tag{3.4}$$

equation (3.2) is fulfilled trivially for any wave function. So indistinguishability by indices does not lead to any requirement on the form of the wave function.

On the other hand, when the particles are thought to be distinguishable by their indices, we have no reason to expect equality of, e.g., the expectation values $\langle \mathbf{r}_1 \rangle$ and $\langle \mathbf{r}_2 \rangle$ of the position variables of two identical particles 1 and 2 for all possible states of the system. As we saw in Sec. 2 it is possible then to prepare states in which the particles 1 and 2 are in disjoint regions of space. It follows that in this case the requirement (3.2) is not legitimate. So, also from this point of view (anti)symmetrization of the wave function is not invoked.

A derivation of (3.4) is given by Schweber (1961), starting again from (3.2), which is required by indistinguishability of the particles. It is interesting to note that (3.4) only follows from (3.2) when this last equality should hold for *all* functions $\psi(x_1, \ldots, x_n)$. When only symmetric or antisymmetric wave functions would play a role in the physics of identical particles, it would be sufficient to consider (3.2) for states corresponding to this class of functions only. But under this circumstance (3.2) is satisfied for any observable $f(x_1, \ldots, x_n)$. So, in order to arrive at (3.4) as a necessary condition the observables of a system of identical particles should obey, it is essential that non- (anti)symmetric wave functions have experimental pertinence. But this implies that the whole reasoning breaks down because, as we saw in Sec. 2, in this case the indices may have an experimentally relevant distinguishing function.

We arrive at the conclusion that a fundamental relationship between "statistics" and indistinguishability by means of indices of identical particles is not evident.

Note that this conclusion is not at variance with the example given in Sec. 2, where departure from "statistics" involved a kind of distinguishability. As a matter of fact, this distinguishability (the Schiff version) was arrived at through position measurement, not through direct observation of some physical property corresponding to the index. The one-to-one correspondence between position and index obtaining here is a very special circumstance, not liable to be a general issue: Deviation from "statistics" is not bound to imply this distinguishability by indices. The intermediate states in the diffusion process described in Section 2 might furnish illustrations to support this view, provided the state function (or density operator) of the system evolves in a continuous fashion from initial to final state also in this case.

"Statistics" not being related to (in)distinguishability, it should be possible to describe this process using either an indexed- or an unindexed-particle language. However, viewed as a diffusion process, an indexed-particle language may describe it in a conceptually simpler manner than an unindexed-particle language. This is the case because now (anti)symmetry of the wave function of, e.g., two identical particles can be interpreted in a simple way by stating that, after the diffusion process has reached its state of equilibrium, the equality

$$|\psi(x_a, x_b)|^2 = |\psi(x_b, x_a)|^2$$

expresses the equality of the probability of finding particles 1 and 2 with coordinates x_a and x_b, respectively, and the probability of finding the particles with their positions and spins exchanged.

By the diffusion process an initial state of two uncorrelated systems [e.g., with density operator (2.2)] is turned over into a final (equilibrium) state in which the systems are correlated. It is preferable to describe this correlation in the unindexed language, i.e., by means of the commutation relations of the field operators of quantum field theory, because it is not a correlation of the *indices* (which seem to be distributed as randomly as possible in the equilibrium state), but of the probabilities of finding particles in different points of space (cf. Sec. 1). However, the theories being equivalent, the correlation should also be expressible in the indexed language.

A fundamental relationship between "statistics" and indistinguishability by means of indices not being evident, we have to consider the tendency of an assembly of identical particles to develop its state into a state showing a specific kind of correlation (either Bose-Einstein or Fermi-Dirac "statistics"), as an independent and fundamental property of the particles. Approaching "statistics" in this way offers a big advantage in understanding in statistical mechanics the factor $N!$ by which the a priori probabilities should be divided in order to get the (correct) Boltzmann weights (e.g., §4.2 of ter Haar, 1961). It is often stated that this correction is really a quantum mechanical effect because only here would indistinguishability of identical particles have an experimental meaning (however, see Rushbrooke, 1949). In quantum mechanics the factor $N!$ turns out to be a consequence of the restriction to (anti)symmetric states, i.e. to *correlated* states. But correlation is not a typically quantum mechanical notion. We have no reason to suppose that this property gets lost in the classical limit. So we have the possibility of attaining the factor $N!$ also classically, when we postulate also for classical mechanics that identical particles are necessarily *correlated* particles (at least at equilibrium). In fact this is equivalent to the replacement already performed by Gibbs, of "specific phases" by "generic phases." (Gibbs, 1902). No allowance has to be made for indistinguishability of states under exchange of particles in phase space when this "statistics" correlation is considered also in classical mechanics as a fundamental property of systems of identical particles. Apart from procuring the factor $N!$ in classical mechanics, "statistics" correlation of the particles does not seem to have consequences. This is illustrated by the vanishing in the classical limit of all interference terms brought forth by (anti)symmetry of the wave functions, making expection values coincide with those obtained from unsymmetrized wave functions. So it is possible to continue using the conventional classical mechanics of uncorrelated particles, as the "statistics" correlation manifests itself only on the quantum level.

4. *A Field Theory of Indexed Particles*

Elementary quantum mechanics is not unconditionally the indexed-particle version of quantum field theory, since particle creation and annihilation is not described by the first theory. In this section we will present a *field* theory of *indexed* particles, i.e., a theory by which it is possible to describe creation and annihilation of indexed particles. In order to perform this extension of quantum field theory, we start from the very clear formulation of this theory given by Robertson (1973). In the unindexed-particle theory a system of N bosons or fermions is represented by a state vector in Fock space $|\Psi_N\rangle$ which is related to the (anti)symmetric wave function $\Psi_s(x_1, \ldots, x_N)$ through the formula

$$|\Psi_N\rangle = (N!)^{-1/2} \int dx_1 \cdots \int dx_N \Psi_s(x_1, \ldots, x_N) \psi^\dagger(x_N) \cdots \psi^\dagger(x_1)|o\rangle \tag{4.1}$$

Here $\psi^\dagger(x)[\psi(x)]$ is the usual creation (annihilation) operator of an unindexed particle at x, obeying the commutation relations for bosons c.q. fermions

$$[\psi(x), \psi(x')]_\mp = 0 \tag{4.2a}$$

$$[\psi(x), \psi^\dagger(x')]_\mp = \delta(x - x') \tag{4.2b}$$

$|o\rangle$ is the vacuum state, defined by

$$\psi(x)|o\rangle = 0 \tag{4.3}$$

It is shown by Robertson (1973) that the wave function $\Psi_s(x_1, \ldots, x_N)$ is obtained from the Fock state (4.1) through

$$\Psi_s(x_1, \ldots, x_N) = (N!)^{-1/2} \langle o|\psi(x_1) \cdots \psi(x_N)|\Psi_N\rangle \tag{4.4}$$

We point out here that (4.1) is not a unique expression for the state vector $|\Psi_N\rangle$. We may replace the function $\Psi_s(x_1, \ldots, x_N)$ in the integrand of (4.1) by an arbitrary function $\Psi(x_1, \ldots, x_N)$ obeying

$$\Psi_s(x_1, \ldots, x_N) = (1/N!) \sum_P \epsilon^P \Psi(P(x_1, \ldots, x_N)) \tag{4.5}$$

the summation being performed over all permutations of $(x_1, \ldots, x_N)$; for bosons $\epsilon^P = +1$, for fermions $\epsilon^P = +1$ or -1 when the permutation P is even or odd. The result (4.4) is independent of the specific form of $|\Psi_N\rangle$ as long as $\Psi(x_1, \ldots, x_N)$ obeys (4.5).

It is not possible to construct a complete description of indexed particles using just a single field operator $\psi(x)$. When all particles have different indices, a different field operator $\psi_i(x)$ has to be associated with each particle. We will call this operator an indexed field operator. The vacuum state, which is as usual defined by

$$\psi_i(x)|o\rangle = 0, \quad \forall_{i\in I} \tag{4.6}$$

is the direct product of the vacuum states $|o_i\rangle$ of all particles contained in the system under study (index set I).

We will restrict our attention here to systems in which the particles are all characterized by different indices, although this may turn out to be too restrictive, as is shown by the example of Sec. 2, where all particles of subsystem A or B may be thought to have the same index. However, field operators $\psi_A(x)[\psi_B(x)]$ describing the fields of the A(B) particles are related to the one-particle field operator $\psi_i(x)$ in the same way as the unindexed field operators of boson and fermion fields. Since in Sec. 6 we shall find the relation between indexed and unindexed field operators, we do not need to study many-particle field operators like $\psi_A(x)$, which moreover seem to be less fundamental from the point of view of distinguishability.

We define the state vector corresponding to a system of N differently indexed particles with wave function $\Psi_s(x_1, \ldots, x_N)$ by analogy of (4.1) as

$$|\Psi_{1,\ldots,N}\rangle = \int dx_1 \cdots \int dx_N \Psi_s(x_1, \ldots, x_N)\psi_N^\dagger(x_N) \cdots \psi_1^\dagger(x_1)|o\rangle \tag{4.7}$$

We also require that the wave function is found from $|\Psi_{1,\ldots,N}\rangle$ through

$$\Psi_s(x_1, \ldots, x_N) = \langle o|\psi_1(x_1) \cdots \psi_N(x_N)|\Psi_{1,\ldots,N}\rangle \tag{4.8}$$

In (4.1) the "statistics" correlation of the particles is incorporated in the state vector in two ways: by the commutation relations of the field operators *and* by the symmetry character of the function $\Psi_s(x_1, \ldots, x_N)$. We have seen that the first one is sufficient to give the right "statistics." In the formulation we develop in this section we will choose the second manner, i.e., we introduce "statistics" in the state vector (4.7) by means of the symmetry character of the function $\Psi_s(x_1, \ldots, x_N)$. By doing so the possibility is created of a uniform description, independent of "statistics," of the indexed particles, be it bosons, fermions, particles obeying parastatistics, or even particles with no correlation at all. The commutation relations of the field operators $\psi_i(x)$, etc. may be the same in all cases.

Since for uncorrelated particles no symmetry requirements are to be imposed on the function $\Psi_s(x_1, \ldots, x_N)$ in (4.7) and (4.8), it follows that

$$\langle o|\psi_1(x_1) \cdots \psi_N(x_N)\psi_N^\dagger(y_N) \cdots \psi_1^\dagger(y_1)|o\rangle = \prod_{i=1}^{N} \delta(x_i - y_i) \tag{4.9}$$

Another aspect of applicability of the theory to uncorrelated particles is expressible by the property that indexed field operators with different indices commute. Thus

$$[\psi_i(x), \psi_j(y)]_- = [\psi_i(x), \psi_j^\dagger(y)]_- = 0, i \neq j \tag{4.10}$$

It incidentally follows from (4.6) and (4.10) that

$$\psi_i(x)|\Psi_{1,\ldots,N}\rangle = 0, \quad i \notin \{1, \ldots, N\} \tag{4.11}$$

which is a most desirable result.

A third requirement for the indexed field operators is suggested by a relation that is equally valid for bosons as well as fermions, viz.,

$$\psi(x)\psi^\dagger(y)|o\rangle = \delta(x-y)|o\rangle$$

As this clearly is independent of "statistics", we expect that it should hold also for our indexed field operators. So we require

$$\psi_i(x)\psi_i^\dagger(y)|o\rangle = \delta(x-y)|o\rangle \tag{4.12}$$

As the equality (4.9) is a direct consequence of (4.10) and (4.12), we may omit the requirement (4.9). The one- and two-particle operators corresponding to the configuration space operators $T_i(x_i)$ and $V_{ij}(x_i, x_j), i \neq j$ are also defined by analogy with quantum field theory, as

$$T_i = \int dx \psi_i^\dagger(x) T_i(x) \psi_i(x) \tag{4.13a}$$

$$V_{ij} = \int dx \int dy \psi_i^\dagger(x)\psi_j^\dagger(y) V_{ij}(x, y)\psi_j(y)\psi_i(x), \quad i \neq j \tag{4.13b}$$

Note that in quantum field theory, because of the presupposed indistinguishability of the particles, only operators are considered corresponding to observables that are symmetric in the particle indices, such as

$$\sum_{i=1}^{N} T_i(x_i)$$

and

$$\sum_{\substack{i,j=1 \\ (i \neq j)}}^{N} V_{ij}(x_i, x_j)$$

With (4.13a) and (4.13b) it is quite easy to show that (4.10) and (4.12) are sufficient for the equivalence, under the correspondence defined by (4.7) of the configuration space representation [by wave functions $\Psi_s(x_1, \ldots, x_N)$] and the representation by state vectors $|\Psi_{1,\ldots,N}\rangle$. For all kinds of "statistics" this equivalence is expressed by the equalities

$$\langle \Phi_{1,\ldots,N}|\Psi_{1,\ldots,N}\rangle = \int dx_1 \cdots \int dx_N \Phi_s^*(x_1, \ldots, x_N)\Psi_s(x_1, \ldots, x_N) \tag{4.14a}$$

$$\langle \Phi_{1,\ldots,N}|T_i|\Psi_{1,\ldots,N}\rangle = \int dx_1 \cdots \int dx_N \Phi_s^*(x_1, \ldots, x_N) T_i(x_i) \Psi_s(x_1, \ldots, x_N), \quad i \in \{1, \ldots, N\} \tag{4.14b}$$

$$\langle\Phi_{1,\ldots,N}|V_{ij}|\Psi_{1,\ldots,N}\rangle = \int dx_1 \cdots \int dx_N \Phi_s^*(x_1,\ldots,x_N) V_{ij}(x_i,x_j)\Psi_s(x_1,\ldots,x_N), \quad i,j \in \{1,\ldots,N\} \tag{4.14c}$$

From (4.11) it is immediately seen that $T_i|\Psi_{1,\ldots,N}\rangle$ and $V_{ij}|\Psi_{1,\ldots,N}\rangle$ vanish when particle i or j is not present in the system described by $|\Psi_{1,\ldots,N}\rangle$. Also, the equivalence of the Schrödinger equation of a system of N indexed identical particles

$$H(1,\ldots,N)|\Psi_{1,\ldots,N}\rangle = i\hbar(d/dt)|\Psi_{1,\ldots,N}\rangle \tag{4.15a}$$

$$\begin{aligned} H(1,\ldots,N) &= H_0(1,\ldots,N) + V(1,\ldots,N) \\ &= \sum_{i=1}^{N} \int dx \psi_i^\dagger(x)\left(-\frac{\hbar^2}{2M}\Delta + V(x)\right)\psi_i(x) \\ &+ \frac{1}{2} \sum_{\substack{i,j=1 \\ i=j}}^{N} \int dx \int dy \psi_i^\dagger(x)\psi_j^\dagger(y)V(x,y)\psi_j(y)\psi_i(x) \end{aligned} \tag{4.15b}$$

with the usual non-relativistic equation in configuration space, is easily derived.

To conclude this section we mention some further properties of the indexed field operators. From the physical interpretation of the operator $\psi_i(x)$ as an operator describing a field of only *one* particle, it follows that

$$\psi_i(x)\psi_i(y) = 0 \tag{4.16}$$

In order to study the commutator of $\psi_i(x)$ and $\psi_i^\dagger(y)$ we switch in the usual manner to a discrete representation:

$$\psi_i(x) = \sum_k a_i^k u_k(x) \tag{4.17}$$

where $\{u_k(x)\}$ is some complete orthonormal set of one-particle functions. Then a_i^k ($a_i^{k\dagger}$) is the annihilation (creation) operator of particle i in the state with wave function $u_k(x)$. Inserting (4.17) into (4.10), (4.12), and (4.16) we get

$$[a_i^k, a_j^l]_- = [a_i^k, a_j^{l\dagger}]_- = 0, \quad i \neq j \tag{4.18a}$$

$$a_i^k a_i^{l\dagger}|o\rangle = \delta_{kl}|o\rangle \tag{4.18b}$$

$$a_i^k a_i^l = 0 \tag{4.18c}$$

When we introduce for the one-particle state of particle i with wave function $\Psi_s(x) = u_k(x)$ the notation

$$|k_i\rangle = a_i^{k\dagger}|o\rangle \tag{4.19}$$

we conclude from (4.18b) that a_i^k may be represented by

$$a_i^k = |o_i\rangle\langle k_i| \tag{4.20}$$

From (4.20) we easily derive that

$$[\psi_i(x), \psi_i^\dagger(y)]_- = \delta(x-y)|o_i\rangle\langle o_i| - \psi_i^\dagger(y)|o_i\rangle\langle o_i|\psi_i(x) \tag{4.21}$$

and

$$[\psi_i(y), \psi_i^\dagger(x)\psi_i(x)]_- = \delta(x-y)\psi_i(x) \tag{4.22}$$

which may be used to find the equation of motion of $\psi_i(x)$ in the Heisenberg picture with the Hamiltonian (4.15b) as

$$i\hbar \frac{\partial \psi_i(x)}{\partial t} = \left[-\frac{\hbar^2}{2M}\Delta + V(x)\right]\psi_i(x) + \tfrac{1}{2}\psi_i(x) \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \int dy\,[V(x,y)+V(y,x)]\,\psi_j^\dagger(y)\psi_j(y) \tag{4.23}$$

It is also easily verified that the operator

$$N_i = \int dx\,\psi_i^\dagger(x)\psi_i(x) = \sum_k a_i^{k\dagger} a_i^k \tag{4.24}$$

commutes with the Hamiltonian (4.15b), expressing conservation of the probability that particle i is present in the system described by this Hamiltonian. The operator corresponding to the number of particles with state function $u_k(x)$ is given by

$$N^k = \sum_{i\in I} a_i^{k\dagger} a_i^k$$

since

$$N^k a_{i_1}^{k_1\dagger} \cdots a_{i_N}^{k_N\dagger}|o\rangle = n_k a_{i_1}^{k_1\dagger} \cdots a_{i_N}^{k_N\dagger}|o\rangle \tag{4.25}$$

n_k being the number of times k is represented in $k_1, \ldots, k_N$. The operators N^k are defined on the Hilbert space of indexed particles, in which the vectors $a_{i_1}^{k_1\dagger} \cdots a_{i_N}^{k_N\dagger}|o\rangle$, $N = 0, 1, 2 \ldots$, $i_j \epsilon I$, $i_j \neq i_k$, constitute an orthonormal basis. The operator corresponding to the total number of particles now follows from (4.24) and (4.25) as

$$\mathcal{N} = \sum_i N_i = \sum_k N^k = \sum_{ik} a_i^{k\dagger} a_i^k \tag{4.26}$$

For the Hamiltonian (4.15b) we arrive in the k representation at the expression

$$H(1,\ldots,N) = \sum_{i=1}^{N} \sum_{kl} H_{kl}^o a_i^{k\dagger} a_i^l + \tfrac{1}{2} \sum_{\substack{i,j=1 \\ (i\neq j)}}^{N} \sum_{klmn} V_{klmn} a_i^{k\dagger} a_j^{l\dagger} a_j^m a_i^n \tag{4.27}$$

$$H_{kl}^o = \int dx\,u_k^*(x)\left[-\frac{\hbar^2}{2M}\Delta + V(x)\right]u_l(x)$$

$$V_{klmn} = \int dx \int dy\,u_k^*(x)u_l^*(y)V(x,y)u_m(y)u_n(x)$$

Mirman (1973) strongly opposes against what is called "physical exchange of particles," in which two particles change places. Indeed, in a theory of indistinguishable particles this notion makes no sense at all as it is then thought to be impossible to distinguish experimentally between the original state and one in which the particles have been exchanged. In a theory of indistinguishable as well as distinguishable particles an interaction is needed to perform the exchange. However, in an indexed-particle theory a new possibility opens up for interpreting the "physical exchange of particles" that avoids the difficulties connected with the more or less mysterious way in which two material particles should be able to exchange their places. For, if not the particles themselves, but just their *indices* would exchange, the final state could not be distinguished experimentally from a state that results from a real physical exchange of the particles, included indices. So an interaction that exchanges just the indices could be interpreted as an exchange interaction.

However, when index exchanging interactions are present it is no longer possible to use this index for distinguishing purposes. As a matter of fact precisely the presence of this kind of interaction would give the index the status of a dynamical variable. So a theory of distinguishable particles is possible only when the interactions are index preserving. The Hamiltonian (4.15b) clearly is of this kind as

$$\langle o|\psi_i(x)\psi_j(y)V(1,\ldots,N)\psi_j^\dagger(x)\psi_i^\dagger(y)|o\rangle = 0, \quad x \neq y$$

5. *Indexed Boson Operators*

In order not to be bothered by phase conventions, in the following we restrict our considerations to bosons. Using the indexed particle operators defined in (4.18) and (4.20) a normalized symmetric state of N indexed bosons may be represented by

$$|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s = \{N!\Pi_k n_k!\}^{-1/2} \sum_P a_{i_1}^{Pk_1\dagger} \cdots a_{i_N}^{Pk_N\dagger}|o\rangle \quad (5.1)$$

Again n_k is the number of times the single particle state k occurs among $k_1, \ldots, k_N$; further $Pk_1, \cdots, Pk_N$ stands for the permutation $P(k_1, \cdots, k_N)$. It is possible (Jauch, 1966) to define a projection operator on the Fock space of symmetric states, Π_s, according to

$$\Pi_s a_{i_1}^{k_1\dagger} \cdots a_{i_N}^{k_N\dagger}|o\rangle = (N!)^{-1/2}(\Pi_k n_k!)^{1/2}|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s \quad (5.2)$$

This projection operator is expressible explicitely in the indexed field operators following

$$\Pi_s = \sum_{N=0}^{\infty} \sum_{i_1 \cdots i_N} \sum_{\{n_k\}} (N!\Pi_k n_k!)^{-1} \sum_{P,Q} a_{i_1}^{Pk_1\dagger} \cdots a_{i_N}^{Pk_N\dagger}|o\rangle\langle o|a_{i_N}^{Qk_N} \cdots a_{i_1}^{Qk_1} \quad (5.3)$$

In (5.3) the summation over $i_1, \ldots, i_N$ is extending over all possible sets of N different indices.

The operators

$$_sb_i^k = \Pi_s a_i^k \Pi_s \tag{5.4a}$$

$$_sb_i^{k\dagger} = \Pi_s a_i^{k\dagger} \Pi_s \tag{5.4b}$$

are easily verified to possess properties that are analogous to those of the usual annihilation and creation operators of unindexed bosons, viz.,

$$|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s = (N!)^{1/2}(\Pi_k n_k!)^{-1/2} {}_sb_{i_1}^{k_1\dagger} \cdots {}_sb_{i_N}^{k_N\dagger}|o\rangle \tag{5.5}$$

$$_sb_i^k|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s = p_i n_k{}^{1/2} N^{-1/2} |i_1 \cdots \not{i} \cdots i_N; k_1 \cdots \not{k} \cdots k_N\rangle_s \tag{5.6a}$$

$$_sb_i^{k\dagger}|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s$$
$$= (1 - p_i)(n_k + 1)^{1/2}(N + 1)^{-1/2}|i_1 \cdots i_N i; k_1 \cdots k_N k\rangle_s \tag{5.6b}$$

where

$$p_i = \begin{cases} 1, i \in \{i_1 \cdots i_N\} \\ 0, i \notin \{i_1 \cdots i_N\} \end{cases}$$

The equalities (5.5) and (5.6) lend support to the interpretation of the operators $_sb_i^k$ and $_sb_i^{k\dagger}$ as annihilation and creation operators of an indexed boson. Contrary to the creation and annilation operators defined in Sec. 4, they pertain to particles correlated according to "statistics." Yet they have a wider scope then the creation and annihilation operators of unindexed field theory, as the operators (5.4) are also defined outside the Fock space of symmetric states. However, since they are adapted especially to this space, it is questionable whether a physical meaning can be attributed to these operators outside the Fock space of symmetric states. For this reason we will only study their behavior on this latter space. The commutation relations of the operators $_sb_i^k$ and $_sb_i^{k\dagger}$ are easily derived from (5.6) in this case. We get

$$[_sb_i^k, {}_sb_j^l]_- = [_sb_i^{k\dagger}, {}_sb_j^{l\dagger}]_- = 0 \ \forall_{ijkl} \tag{5.7}$$

which relations are even valid outside the Fock space of symmetric states. The commutation relation $[_sb_i^k, {}_sb_j^{l\dagger}]_-$ is rather a bulky expression which we shall not write down here.

In order to arrive at a dynamical description of a system of indexed bosons by means of the indexed boson operators, we mention the equality

$$a_i^k|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s = {}_sb_i^k|i_1 \cdots i_N; k_1 \cdots k_N\rangle_s \tag{5.8}$$

(note that there does not exist an analogous equality for the creation operators). Replacing in (4.27) the operators a_i^k etc. by their symmetrized counterparts, we may define the Hamiltonian operator

$$_sH(1, \ldots, N) = \sum_{i=1}^{N} \sum_{kl} H_{kl}^0 \, {}_sb_i^{k\dagger} \, {}_sb_i^l + \tfrac{1}{2} \sum_{\substack{i,j=1 \\ (i \neq j)}} \sum_{klmn} V_{klmn} \, {}_sb_i^{k\dagger} \, {}_sb_j^{l\dagger} \, {}_sb_j^m \, {}_sb_i^n \tag{5.9}$$

With (5.8) it is then easily seen that

$$\begin{aligned} & {}_s\langle 1 \cdots N; k_1 \cdots k_N \,|\, {}_sH(1, \ldots, N) \,|\, 1 \cdots N; l_1 \cdots l_N \rangle_s \\ & = {}_s\langle 1 \cdots N; k_1 \cdots k_N \,|\, H(1, \ldots, N) \,|\, 1 \cdots N; l_1 \cdots l_N \rangle_s \end{aligned} \tag{5.10}$$

So, on the Fock space of symmetric states we have

$${}_sH(1, \ldots, N) = H(1, \ldots, N) \tag{5.11}$$

from which we conclude that (5.9) is the Hamiltonian of a system of N indexed bosons. By the substitution $a_i^k \to {}_sb_i^k$ also the number operators defined by (4.24), (4.25), and (4.26) are turned over into the corresponding number operators of indexed bosons.

The operator ${}_sb_i^{k\dagger}$ is, unlike $a_i^{k\dagger}$, not simply interpretable as the creation operator of a particle with index i *in* single particle state k. This is most clearly illustrated by the equality

$${}_sb_i^{k\dagger}\, {}_sb_j^{l\dagger} = {}_sb_j^{k\dagger}\, {}_sb_i^{l\dagger} \tag{5.12}$$

which follows from the symmetry of the vectors (5.5). Equation (5.12) reflects the facts that bosons are correlated particles and that the notion of single particle states has a limited meaning for such particles. For, although the operator ${}_sb_i^{k\dagger}$ adds a particle with index i and a single-particle state k to the initial state, because of the correlation in the final state particle i is not bound to be found in state k.

6. *Relation to Unindexed Boson Theory*

In order to be able to compare the indexed boson formalism with the usual formalism of "indistinguishable," unindexed bosons (4.1)-(4.3) it is necessary to remove the restriction in (5.9) on the summations of i and j to a specified number of particles. We therefore replace (5.9) by the Hamiltonian

$${}_sH = \sum_i \sum_{kl} H_{kl}^0\, {}_sb_i^{k\dagger}\, {}_sb_i^l + \tfrac{1}{2} \sum_{\substack{i,j \\ (i \neq j)}} \sum_{klmn} V_{klmn}\, {}_sb_i^{k\dagger}\, {}_sb_j^{l\dagger}\, {}_sb_j^m\, {}_sb_i^n \tag{6.1}$$

where the summations of i and j now extend over the whole set I of the particle indices. The Hamiltonian (6.1) has the desirable property that its matrix elements between state vectors not belonging to a definite number of particles, of the form

$$|\Psi\{i\}\rangle = \sum_N \sum_{\substack{\{nk\} \\ (\Sigma n_k = N)}} c_{\{nk\}}^{i_1 \cdots i_N} |i_1 \cdots i_N; k_1 \cdots k_N\rangle_s \tag{6.2}$$

are equal to those of the unindexed boson Hamiltonian

$$H = \sum_{kl} H_{kl}^0 a_k^\dagger a_l + \tfrac{1}{2} \sum_{klmn} V_{klmn} a_k^\dagger a_l^\dagger a_m a_n \tag{6.3}$$

between state vectors

$$|\Psi\rangle = \sum_{\{n_k\}} c\{nk\}|\{n_k\}\rangle \tag{6.4}$$

provided corresponding coefficients in (6.2) and (6.4) are identified, i.e., when we put

$$c\{n_k\} = c^{i_1 \cdots i_N}_{\{n_k\}} \tag{6.5}$$

then

$$\langle\Psi|H|\Phi\rangle = \langle\Psi_{\{i\}}|_s H|\Phi_{\{i\}}\rangle \tag{6.6}$$

We note that the equality of the matrix elements of $_sH$ with those of H is independent of the choice of the indices in (6.2) as long as (6.5) is satisfied. A second remark amounts to the observation that the state vectors (6.2) do not contain linear superpositions of states with an equal number of particles but having different indexations. This restriction is reasonable within the theory expounded here, since the indices are considered to be intrinsic properties of the particles and so there should obtain a superselection rule regarding these properties. This aspect is very important in tracing the relation between the indexed and unindexed operators a_i^k and a_k. Indeed, on behalf of this superselection rule an identification of a_k with $\Sigma_i a_i^k$ as is suggested by (6.1) is ruled out. Apart from this argumentation, introduction of this identification in (6.3) gives rise to matrix elements of which the potential energy part differs from that given by (6.1) by a factor of 2. It may also be verified that the matrix elements of $\Sigma_i a_i^k$ do not equal those of the unindexed boson operator calculated in states connected by (6.5), but that the matrix elements of the operator $\Sigma_i a_i^k \mathcal{N}^{1/2}$ do (analogously for the creation operators). However, the physical meaning of this operator is not clear.

From the foregoing it follows that the results of "indistinguishable" unindexed boson theory should be recovered by application of classical ensemble theory as far as the indices are concerned. To this end we introduce in the Hilbert space of symmetric indexed particle vectors (6.2) equivalence classes of state vectors in such a way that state vectors differing only by the indexation of the particles belong to the same class. Such a class is then represented by the unindexed state vector (6.4), $c_{\{n_k\}}$ being given by (6.5). A transition between two unindexed particle states $|\Psi\rangle$ and $|\Phi\rangle$, specified by the transition amplitude $\langle\Phi|\Psi\rangle$ is now represented in the indexed-particle formalism by a transition between the corresponding equivalence classes, specified by the amplitude $\langle\Phi_{\{i\}}|\Psi_{\{i\}}\rangle$ averaged over the different possible indexations within the equivalence classes.

Because of the different meanings of the operators a_i^k and a_k it is not possible to recover the matrix elements $\langle\Psi|a_k|\Phi\rangle$ by simply averaging the corresponding indexed-particle matrix elements. Indeed, in interpreting the annihilation of an unindexed particle in terms of indexed particles whose indices remain unobserved, we should realize that each of the particles that are present in the initial state has a probability of being annihilated. The sum of these probabilities should equal the probability that a particle (no matter what is its index) is annihilated. This picture is in agreement with the equality

$$
\begin{aligned}
&|\langle \{n_k\}', n_k - 1 | a_k | \{n_k\}\rangle|^2 = n_k = N(n_k/N) \\
&= \sum_i |_s\langle i_1 \cdots \not{i} \cdots i_N; k_1 \cdots \not{k} \cdots k_N | a_i^k | i_1 \cdots i_N; k_1 \cdots k_N\rangle_s|^2
\end{aligned}
\tag{6.7}
$$

A similar argument obtains for creation operators and for arbitrary products $p(\{a_k^\dagger\}, \{a_l\})$ of creation and annihilation operators.

Thus for instance for $k \neq l$

$$
\begin{aligned}
&|\langle\{n_k\}', n_k - 1, n_l - 1 | a_k a_l | \{n_k\}\rangle|^2 = n_k n_l = N(N-1)\frac{n_k n_l}{N(N-1)} \\
&= \sum_{i,j} |_s\langle i_1 \cdots \not{i} \cdots \not{j} \cdots i_N; k_1 \cdots \not{k} \cdots \not{l} \cdots k_n | a_i^k a_j^l | i_1 \cdots i_N; k_1 \cdots k_N\rangle_s|^2
\end{aligned}
\tag{6.8}
$$

Here $N(N-1)$ is the number of different ways in which two indexed particles may be annihilated from a system of N indexed particles. Note that the order in which particles i and j are annihilated is relevant, giving rise to two different transition processes.

It does not seem possible to relate directly the *amplitudes* of the indexed- and the unindexed-particle theories in a way consistent with the interpretation given above, because of the appearance of the factors $N^{-1/2}$ expounded in (5.6). Since amplitudes do not have physical relevance, this does not have consequences in the cases studied above. However, when we try to generalize the foregoing to matrix elements between arbitrary state vectors, we arrive at the conclusion that this can only be done when one of the vectors has a definite number of particles. Thus

$$
|\langle\Psi_N | P(\{a_k^\dagger\}, \{a_l\}) | \Phi\rangle|^2 = \sum_{\{r\},\{s\}} |\langle\Psi_{i_1} \cdots {}_{i_N} | P(\{a_r^{k\dagger}\}, \{a_s^l\}) | \Phi_{\{i\}}\rangle|^2
\tag{6.9}
$$

When both vectors have an indefinite number of particles the indexed-particle probabilities differ from the unindexed-particle ones as may be seen from the following simple example. With

$$
|\Phi\rangle = a_1 |\{n_k\}\rangle + b_1 |\{m_k\}\rangle, \quad |\Psi\rangle = a_2 |\{n_k\}', n_k - 1\rangle + b_2 |\{m_k\}', m_k - 1\rangle
$$

$$
N = \sum_k n_k < M = \sum_k m_k
$$

we get

$$
|\langle\Phi | a_k | \Psi\rangle|^2 = |a_1^* a_2 \sqrt{n_k} + b_1^* b_2 \sqrt{m_k}|^2
\tag{6.10}
$$

but

$$\sum_j |\langle \Phi_{\{i\}} | a_j^k | \Psi_{\{i\}} \rangle|^2 = N |a_1^* a_2 \sqrt{(n_k/N)} + b_1^* b_2 \sqrt{(m_k/M)}|^2$$
$$+ (M - N) |b_1^* b_2 \sqrt{(m_k/M)}|^2 \qquad (6.11)$$

which is different from (6.10) when $M \neq N$, the difference being proportional to $(N/M)^{1/2} - 1$.

The discrepancy between the indexed- and unindexed-particle theories found here could be used to test the applicability of the theories. For this it is essential to consider transitions in which both initial and final states are coherent superpositions of states with different numbers of particles.

7. *Comparison with an Earlier Theory*

By Marx (1972) creation and annihilation operators have been introduced of "numbered" bosons and fermions. The creation operator numbered j is defined in such a way that the (anti)symmetric state vector in (generalized) Fock space containing particles numbered $1, 2, \ldots, j-1, j+1, \ldots, N$ is turned over into a state vector of the same symmetry character with particles numbered $1, 2, \ldots, N$. Analogously the annihilation operator numbered j destroys a particle with this same number, maintaining the symmetry character of the state vector. In view of (5.6) the creation and annihilation operators of numbered particles show at first sight some resemblance to the indexed-particle operators defined by (5.4). The numbered particle operators of Marx refer to correlated particles, as do the operators ${}_s b_i^{k\dagger}$ and ${}_s b_i^k$. Apart from this similarity the two kinds of operators are quite different because of the different meanings of "numbering" and "indexation" of particles. These differences are for instance reflected by the commutation relations of the operators, which show quite a different behavior.

In Section 2 we defined the particle indices to be intrinsic properties of the particles. As we saw at the end of Sec. 5, we then cannot interpret the operator ${}_s b_i^{k\dagger}$ as the creation operator of a particle with index i in single particle state k. As Marx defines his numbered particle operators precisely according to this interpretation, we may conclude that his numbers cannot be intrinsic properties of the particles. This conclusion is confirmed by the commutation relations the numbered particle operators obey [Marx, 1972; Equations (2.6)–(2.10)]. These relations show a dependence of the numbering of the particles on the order in which two particles are created or destroyed successively in two different single-particle states. This could never be the case if the numbers represented intrinsic properties of the particles. In fact, the results of the numbered-particle theory of Marx are compatible with an interpretation of the numbering as a numbering of *occupied* single-particle states, rather than as an indexation of the particles. This means that "particle j" refers to the particle that is attributed to the j^{th} *occupied* single-particle state. Thus, when a particle is added in some single-particle state, all particles that were already present in states with a higher ordinal number should be renumbered $j \to j + 1$. This clearly shows the difference between particle numbering and indexation.

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